Integrals of Motion on Extremals of the Equation Euler - Lagrange

VP Koshcheev[*]

Moscow Aviation Institute (National Research University), Strela Branch, Zhukovsky, Moscow Region, Russia

[*]Corresponding Author

Moscow Aviation Institute (National Research University), Strela Branch, Zhukovsky, Moscow Region, Russia, Tel.: 9854761619, E-mail: koshcheev1@yandex.ru

Abstract

It is shown that a chain of closed systems of first-order ordinary differential equations describing the evolution of moments can be constructed using the Jacobi equation. It is shown that Wronsky determinants for fundamental matrices of closed systems of first–order ordinary differential equations are integrals of motion on extremals of the Euler-Lagrange equation. It is shown that the distribution function is localized in the vicinity of a classical trajectory, the size of which in the phase space is of the order of Planck's constant.

Keywords: Euler-Lagrange equation; Jacobi equation; chain of closed systems of first-order ordinary differential equations; integrals of motion on extremals

It was shown in [1] that a chain of closed systems of first-order ordinary differential equations describing the evolution of moments can be constructed using the Jacobi equation.

Let us be given the Euler–Lagrange equation

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \quad (1)$$

Where $L = L(t, x, \dot{x})$ is the Lagrange function; $\dot{x} = \dfrac{dx}{dt}$.

Following [2], we write

$$L(t, x + \delta x, \dot{x} + \delta\dot{x}) = L(t, x, \dot{x}) + \delta L,$$

$$Where \delta L = \left[\frac{d}{d\alpha}L(t, x + \alpha\delta x, \dot{x} + \alpha\delta\dot{x})\right]_{\alpha=0} = \frac{\partial L(t, x, \dot{x})}{\partial x}\delta x + \frac{\partial L(t, x, \dot{x})}{\partial \dot{x}}\delta\dot{x} \quad (2)$$

Substitute (2) into (1) and obtain the Jacobi equation.

$$\frac{\partial \delta L}{\partial x} - \frac{d}{dt}\frac{\partial \delta L}{\partial \dot{x}} = 0,$$

which is written as [2]

$$\left[L_{x\dot{x}} - \frac{dL_{x\dot{x}}}{dt}\right]\delta x - \frac{d}{dt}(L_{\dot{x}\dot{x}}\delta\dot{x}) = 0 \quad (3)$$

If $L = \dfrac{m\dot{x}^2}{2} - U(x, t)$, then the Euler–Lagrange and Jacobi equations have the form, respectively.

$$U_x + m\ddot{x} = 0 \quad (4)$$

$$U_{xx}\delta x + m\delta\ddot{x} = 0 \quad (5)$$

It can be seen that using the Jacobi equation (5), closed systems of first-order differential equations can be constructed for three new dynamic variables.

$$\frac{d}{dt}\begin{pmatrix} \overline{\delta x^2} \\ \overline{\delta x \delta \dot{x}} \\ \overline{\delta \dot{x}^2} \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 \\ -\dfrac{U_{xx}}{m} & 0 & 1 \\ 0 & -\dfrac{2U_{xx}}{m} & 0 \end{pmatrix}\begin{pmatrix} \overline{\delta x^2} \\ \overline{\delta x \delta \dot{x}} \\ \overline{\delta \dot{x}^2} \end{pmatrix}, \qquad (6)$$

where $\quad \overline{\delta x^2} = (\delta x)^2; \quad \overline{\delta \dot{x}^2} = (\delta\dot{x})^2; \quad \overline{\delta x\,\delta\dot{x}} = (\delta x\,\delta\dot{x}).$

The recurrent formula has the form

$$\frac{d}{dt}\overline{\delta x^k \delta\dot{x}^s} = -s\frac{U_{xx}}{m}\overline{\delta x^{k+1}\delta\dot{x}^{s-1}} + k\overline{\delta x^{k-1}\delta\dot{x}^{s+1}}; \qquad (7)$$

$$k + s = n.$$

$$
\begin{cases}
\dfrac{d}{dt}\,\overline{\delta\hat{x}^2} = \dfrac{2}{m}\,\overline{\delta\hat{x}\delta\hat{p}_x} \\[2mm]
\dfrac{d}{dt}\,\overline{\delta\hat{x}\delta\hat{p}_x} = \dfrac{1}{m}\,\overline{\delta\hat{p}_x^2} - U_{xx}\,\overline{\delta\hat{x}^2} \\[2mm]
\dfrac{d}{dt}\,\overline{\delta\hat{p}_x^2} = -2U_{xx}\,\overline{\delta\hat{x}\delta\hat{p}_x},
\end{cases}
\tag{8}
$$

the initial conditions for which satisfy the Heisenberg uncertainty relations

$$
\begin{aligned}
(\overline{\delta\hat{x}\delta\hat{p}_x})^2\big|_{t=0} &\leq (\overline{\delta\hat{x}^2})\big|_{t=0} \cdot (\overline{\delta\hat{p}_x^2})\big|_{t=0}, \\[2mm]
(\overline{\delta\hat{x}^2})\big|_{t=0} \cdot (\overline{\delta\hat{p}_x^2})\big|_{t=0} &\geq \dfrac{\hbar^2}{4}.
\end{aligned}
\tag{9}
$$

The systems of differential equations (6-7) can be written as [4]

$$
\frac{dW}{dt} = AW,
\tag{10}
$$

where W is the fundamental matrix of solutions, and A is the matrix of coefficients for unknowns.

Using the Jacobi identity [4], it is possible to establish the relationship between the Wronsky determinant at the initial and final moments of time.

$$
|W(t)| = |W(t_0)| \exp\left(\int_{t_0}^{t} \mathrm{Sp}A\,dt\right)
\tag{11}
$$

where SpA 0 is the sum of the diagonal elements of the fundamental matrices of differential equations (6-8) at n 2,3,... zero.

of differential equations (6-8) are integrals of motion on the extremals of the Euler–Lagrange equation (4).

Thus, the ronsky determinants for the fundamental matrices

We construct Liouville equation for the Jacobi equation (5)

$$
\frac{\partial P}{\partial t} + \delta\dot{x}\frac{\partial P}{\partial \delta x} + \frac{U_{xx}}{m}\delta x\frac{\partial P}{\partial \delta\dot{x}} = 0
\tag{12}
$$

The solution to this equation is the distribution function of

the fluctuations of the coordinate and velocity

$$
P(\delta x, \delta\dot{x}, t) = \frac{1}{2\pi\Delta^2}\exp\left(-\frac{(\delta\tilde{x})^2\delta\tilde{p}^2 + (\delta\tilde{x})^2\delta\tilde{p}^2 - 2\langle\delta x\delta\dot{x}\rangle\delta x\delta\dot{x}}{2\Lambda^2}\right)
\tag{13}
$$

where $\quad \Delta^2 = \overline{\delta x^2}\,\overline{\delta\dot{x}^2} - \overline{\delta x\,\delta\dot{x}}^2, \quad$ if the second moments $\overline{\delta x^2}, \overline{\delta\dot{x}^2}, \overline{\delta x\,\delta\dot{x}}$ satisfy the system of equations (6).

Using the system of equations (6), it can be shown that

$$
\frac{d\Delta^2}{dt} = 0
\tag{14}
$$

The system of equations (9) exactly coincides with the system of equations for the mean squares of the quantum fluctuations of the coordinate and momentum [3,4]. Since it follows from formula (10) $\Delta^2(t) = \mathrm{const.}$ for wave functions minimizing the Heisenberg uncertainty ratio, we obtain

$$\left(\overline{\delta x^2}\right)\left(\overline{\delta p_x^2}\right) - \left(\overline{\delta x\,\delta p_x}\right)^2 = \frac{\hbar^2}{4}. \quad (15)$$

Thus, the distribution function (13) is localized in the vicinity of a classical trajectory, the size of which in the phase space is of the order of Planck's constant. Equation (15) determines the extremely small value of the emittance of the charged particle beam $\sim\hbar$, which is important in accelerator technology [5].

References

1. Koshcheev VP (2025) Classical and quantum mechanical corrections to the solution of the Euler– Lagrange equation, The successes of cybernetics, 6: 44-46.

2. IM Gelfand, SV Fomin (1961) Calculus of Variations, Fizmatgiz, Moscow, 1.

3. Bagrov VG, Belov VV, Kandratieva MF (1994) Quasi-classical approximation in quantum mechanics. A new approach, Theoretical and mathematical physics, 98: 48-55.

4. FR Gantmacher (1996) Theory of matrices, Nauka, Moscow, 1.

5. Levichev EB (2018) Electron accumulators with low emittance, Successes of Physical Sciences, 188: 31-54.

SMP Family Medicine and Primary Care

SMP Cardiology and Cardiovascular Medicine

SMP Human Nutrition and Dietetics

SMP Pediatrics and Child Health

SciMed Press Publisher

SMP Journal of Cancer Research

SMP Biotechnology and Bioengineering

SMP Nanotechnology and Nanomedicine

SMP Chemical Engineering Science